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Forms on vector bundles over hyperbolic manifolds and the trace anomaly

A A Bytsenko[1], **E Elizalde**[2,3] **and R A Ulhoa**[1]

[1] Departamento de Fisica, Universidade Estadual de Londrina, Caixa Postal 6001, Londrina-Parana, Brazil

[2] Consejo Superior de Investigaciones Científicas, Institut d'Estudis Espacials de Catalunya (IEEC/CSIC), Edifici Nexus, Gran Capità 2-4, 08034 Barcelona, Spain

[3] Departament ECM, Facultat de Física, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

E-mail: abyts@uel.br and elizalde@ieec.fcr.es

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Abstract

We study gauge theories based on Abelian p-forms on real compact hyperbolic manifolds. An explicit formula for the trace anomaly corresponding to skew-symmetric tensor fields is obtained, by using zeta-function regularization and the trace tensor kernel formula. Explicit exact and numerical values of the anomaly for p-forms of order up to $p = 4$ in spaces of dimension up to $n = 10$ are then calculated.

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1. Introduction

The conformal deformations of the Riemannian metric and the corresponding anomaly play an important role in quantum theories. The role of spacetimes singularity and of (conformal) anomalies in quantum corrections to the energy–momentum tensor is very important in physics. Such anomalies are associated with the scaling character of the effective action [1–3] and, therefore, with the renormalization group. Among other applications of the anomaly, one can mention string theory [4], the c-theorem and its generalizations [5–8], anomaly-induced dynamics in quantum gravity and particle production in a gravitational field [9–11].

It is well known that the evaluation of the anomaly is actually possible for even-dimensional spaces albeit its computation is extremely involved. The general structure of such an anomaly in curved even-dimensional spaces has been actively studied (see, for example, [12]). We briefly mention here an analysis related to this phenomenon for constant curvature spaces. The calculation of the conformal anomaly for the sphere can be found in [13]. Explicit computations of the anomaly (of the stress–energy tensor) for scalar and spinor quantum fields in compact hyperbolic spaces have been carried out in [14, 15] (see also [16, 17]), using the zeta-function regularization method [18–20]. A detailed calculation for the case of spheres can be found in [21, 22].

The purpose of this paper is to analyse the trace anomaly associated with tensor fields on real hyperbolic spaces. We present a decomposition of the Hodge Laplacian and the tensor kernel trace formula associated with free generalized gauge fields (p-forms). The main ingredient required is a type of differential form structure on the physical, auxiliary or ghost variables. We consider spectral functions and the trace anomaly associated with physical degrees of freedom of the Hodge–de Rham operators on p-forms.

The trace anomaly on a conformaly flat geometry (elliptic or hyperbolic geometry, for example) is completely expressed in terms of the Euler characteristic. The coefficient of the Euler characteristic in the trace anomaly is to be viewed as a measure of the degrees of freedom in the field theory, and it should decrease along the renormalization-group flow [23–25].

We calculate the trace anomaly for skew-symmetric tensor fields, and in a dynamical theory, which yield in the end a generalization of the effective action, where such anomalies are to be suppressed. It is important to know explicitly what these terms are, in order to take them into account. We analyse the anomalous behaviour varying the dimensions and making a comparison with conformal invariant scalars. Our results can be used in the anomaly-induced effective action of tensor fields. In particular, anomaly-induced dynamics constitute a fundamental ingredient in order to construct an effective theory of quantum gravity.

2. Exterior forms in hyperbolic spaces

We shall work with an n-dimensional compact real hyperbolic space X with universal covering M and fundamental group Γ. We can represent M as the symmetric space G/K, where $G = SO_1(n, 1)$ and $K = SO(n)$ is a maximal compact subgroup of G. Then we regard Γ as a discrete subgroup of G acting isometrically on M, and we take X to be the quotient space by that action: $X = \Gamma\backslash M = \Gamma\backslash G/K$. Let τ be an irreducible representation of K on a complex vector space V_τ, and consider the induced homogeneous vector bundle $G \times_K V_\tau$ (the fibre product of G with V_τ over K) $\to M$ over M. Factorizing the vector bundle $G \times_K V_\tau$ by the left action of the discrete subgroup Γ, we get a vector bundle $E_\tau \to \Gamma\backslash M = X$ over X. The natural Riemannian structure on M (therefore on X) induced by the Killing form $(,)$ of G gives rise to a connection Laplacian L on E_τ. If Ω_K denotes the Casimir operator of K, that is

$$\Omega_K = -\sum y_j^2 \tag{1}$$

for a basis $\{y_j\}$ of the Lie algebra $\mathfrak{k}_0$ of K, where $(y_j, y_\ell) = -\delta_{j\ell}$, then $\tau(\Omega_K) = \lambda_\tau \mathbf{1}$, for a suitable scalar λ_τ. Moreover, for the Casimir operator Ω of G, with Ω operating on smooth sections $\Gamma^\infty E_\tau$ of E_τ, one has

$$L = \Omega - \lambda_\tau \mathbf{1} \tag{2}$$

(see lemma 3.1 of [26]). For $\lambda \geqslant 0$, let

$$\Gamma^\infty(X, E_\tau)_\lambda = \{s \in \Gamma^\infty E_\tau \mid -Ls = \lambda s\} \tag{3}$$

be the space of eigensections of L corresponding to λ. Here we note that since X is compact we can order the spectrum of $-L$ by taking $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, with $\lim_{j\to\infty} \lambda_j = \infty$. We shall focus on the more difficult (and interesting) case when $n = 2k$ is even, and we specialize τ to be the representation τ_p of $K = SO(2k)$ on $\Lambda^p\mathbb{C}^{2k}$, say $p \neq k$. It is convenient, moreover, to work with the normalized Laplacian $\mathfrak{L} = -c(n)L$, where $c(n) = 2(n-1) = 2(2k-1)$. $\mathfrak{L}$ has spectrum $\{c(n)\lambda_j, m_j\}_{j=0}^\infty$, where the multiplicity m_j of the eigenvalue $c(n)\lambda_j$ is given by

$$m_j = \dim \Gamma^\infty(X, E_{\tau^{(p)}})_{\lambda_j}. \tag{4}$$

Let ω_p, φ_p be exterior differential p-forms. Their invariant inner product is defined by $(\omega_p, \varphi_p) \stackrel{\text{def}}{=} \int_X \omega_p \wedge *\varphi_p$. The following properties for operators and forms hold: $dd = \delta\delta = 0$, $\delta = (-1)^{np+n+1} * d*$, $**\omega_p = (-1)^{p(n-p)}\omega_p$. The operators d and δ are adjoint to each other with respect to this inner product for p-forms: $(\delta\omega_p, \varphi_p) = (\omega_p, d\varphi_p)$. In quantum field theory the Lagrangian associated with ω_p takes the form: $\mathrm{d}\omega_p \wedge *\mathrm{d}\,\omega_p$ (gauge field) , and $\delta\omega_p \wedge *\delta\omega_p$ (co-gauge field). The Euler–Lagrange equations are supplied with the gauge $\delta\omega_p = 0$ (Lorentz gauge), or $d\omega_p = 0$ (co-Lorentz gauge). These Lagrangians give possible representations of tensor fields or generalized Abelian gauge fields. The two representations of tensor fields are not completely independent, because of the well-known duality property of exterior calculus which gives a connection between star-conjugated gauge and co-gauge tensor fields. The gauge p-forms are mapped into the co-gauge $(n-p)$-forms under the action of the Hodge $*$ operator. The vacuum-to-vacuum amplitude for the gauge p-form ω_p becomes [27]:

$$Z = N \int \mathrm{D}\omega \exp[-(\omega, \mathfrak{L}_p\omega)] \prod_{j=1}^{p} (\mathrm{Vol}_{p-j}(\det \mathfrak{L}_{p-j})^{(j+1)/2})^{(-1)^{j+1}} \tag{5}$$

where $\mathfrak{L}_p$ is the Lagrangian on p-forms, and we need to factorize the divergent gauge group volume and integrate over the classes of gauge transformations $(\omega \to \omega + \mathrm{d}\phi)$.

3. The trace formula applied to the tensor kernel

The space of smooth sections $\Gamma^\infty E_\tau$ of E_τ is just the space of smooth p-forms on X. We can therefore apply the version of the trace formula developed by Fried in [28]. First we set up some additional notation. For σ_p the natural representation of $SO(2k-1)$ on $\Lambda^p\mathbb{C}^{2k-1}$, one has the corresponding Harish–Chandra–Plancherel density given—for a suitable normalization of the Haar measure $\mathrm{d}x$ on G—by

$$\mu_{\sigma_p(r)} = \frac{\pi}{2^{4k-4}[\Gamma(k)]^2} \binom{2k-1}{p} P_{\sigma_p}(r) r \tanh(\pi r) \tag{6}$$

for $0 \leqslant p \leqslant k-1$, where

$$P_{\sigma_p}(r) = \prod_{\ell=2}^{p+1} \left[r^2 + \left(k - \ell + \frac{3}{2}\right)^2 \right] \prod_{\ell=p+2}^{k} \left[r^2 + \left(k - \ell + \frac{1}{2}\right)^2 \right] \tag{7}$$

is an even polynomial of degree $2k-2$. One has that $P_{\sigma_p}(r) = P_{\sigma_{2k-1-p}}(r)$ and $\mu_{\sigma_p}(r) = \mu_{\sigma_{2k-1-p}}(r)$ for $k \leqslant p \leqslant 2k-1$. Define the Miatello coefficients [29, 30] $a_{2\ell}^{(p)}$ for $G = SO_1(2k+1, 1)$ by $P_{\sigma_p}(r) = \sum_{\ell=0}^{k-1} a_{2\ell}^{(p)} r^{2\ell}$, $0 \leqslant p \leqslant 2k-1$.

Let $\mathrm{Vol}(\Gamma\backslash G)$ denote the integral of the constant function $\mathbf{1}$ on $\Gamma\backslash G$ with respect to the G-invariant measure on $\Gamma\backslash G$ induced by $\mathrm{d}x$. For $0 \leqslant p \leqslant n-1$ the Fried trace formula applied to the kernel $\mathcal{K}_t = \mathrm{e}^{-t\mathfrak{L}_p}$ holds [28]:

$$\mathrm{Tr}(\mathrm{e}^{-t\mathfrak{L}_p}) = I_\Gamma^{(p)}(\mathcal{K}_t) + I_\Gamma^{(p-1)}(\mathcal{K}_t) + H_\Gamma^{(p)}(\mathcal{K}_t) + H_\Gamma^{(p-1)}(\mathcal{K}_t) \tag{8}$$

where $I_\Gamma^{(p)}(\mathcal{K}_t)$, $H_\Gamma^{(p)}(\mathcal{K}_t)$ are the identity and hyperbolic orbital integrals, respectively,

$$I_\Gamma^{(p)}(\mathcal{K}_t) \stackrel{\text{def}}{=} \frac{\chi(1)\mathrm{Vol}(\Gamma\backslash G)}{4\pi} \int_{\mathbb{R}} \mathrm{d}r\, \mu_{\sigma_p}(r) \exp\left(-t\left(r^2 + p + \rho_0^2\right)\right) \tag{9}$$

$$H_\Gamma^{(p)}(\mathcal{K}_t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma\in C_\Gamma - \{1\}} \frac{\chi(\gamma)}{j(\gamma)} t_\gamma C(\gamma) \chi_{\sigma_p}(m_\gamma) \exp\left(-t\left(\rho_0^2 + p\right) - \frac{t_\gamma^2}{4t}\right). \tag{10}$$

Here $C_\Gamma \subset \Gamma$ is a complete set of representations in Γ of its conjugacy classes, and $C(\gamma)$ is a well-defined function on $\Gamma - \{1\}$ (for more details see [31]), $\rho_0 = (n-1)/2$, and $\chi_\sigma(m) = \mathrm{trace}(\sigma(m))$ is the character σ for $m \in SO(2n-1)$. If $n = 2k$ is even then $\sigma_p (0 \leqslant p \leqslant n-1)$ is always irreducible; if $n = 2k+1$ then every σ_p is irreducible except for $p = (n-1)/2 = k$, in which case σ_k is the direct sum of two spin-$\frac{1}{2}$ representations $\sigma^\pm : \sigma_k = \sigma^+ \oplus \sigma^-$. For $p = k$, the representation τ_k of $K = SO(2k)$ on $\Lambda^k \mathbb{C}^{2k}$ is not irreducible: $\tau_k = \tau_k^+ \oplus \tau_k^-$ is the direct sum of two spin-$\frac{1}{2}$ representations.

The case of the trivial representation. In the case of the trivial representation ($p = 0$, i.e. for smooth functions) the measure $\mu(r) \equiv \mu_0(r)$ corresponds to the trivial representation. Therefore, we take $I_\Gamma^{(-1)}(\mathcal{K}_t) = H_\Gamma^{(-1)}(\mathcal{K}_t) = 0$. Since σ_0 is the trivial representation, one has $\chi_{\sigma_0}(m_\gamma) = 1$. In this case, formula (8) reduces exactly to the trace formula for $p = 0$ [26, 16, 17, 31, 32],

$$I_\Gamma^{(0)}(\mathcal{K}_t) = \frac{\chi(1)\,\mathrm{Vol}\,(\Gamma\backslash G)}{4\pi} \int_{\mathbb{R}} \mathrm{d}r\, \mu_{\sigma_0}(r) \exp\big(-t\big(r^2 + \rho_0^2\big)\big) \tag{11}$$

$$H_\Gamma^{(0)}(\mathcal{K}_t) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_\Gamma - \{1\}} \frac{\chi(\gamma)}{j(\gamma)} t_\gamma C(\gamma) \exp\left(-t\rho_0^2 - \frac{t_\gamma^2}{4t}\right). \tag{12}$$

4. Spectral functions on *p*-forms and the trace anomaly

The transverse part of the skew-symmetric tensor is represented by the co-exact p-form $\omega_p^{(\mathrm{CE})} = \delta\omega_{p+1}$, which trivially satisfies $\delta\omega_p^{(\mathrm{CE})} = 0$, and we denote by $\mathfrak{L}_p^{(\mathrm{CE})}$ the restriction of the Laplacian on the co-exact p-form. The goal now is to extract the co-exact p-form on the manifold which describes the physical degrees of freedom of the system. We get [33–35]

$$\begin{aligned} \mathrm{Tr}\exp\big(-t\mathfrak{L}_p^{(\mathrm{CE})}\big) = \sum_{j=0}^{p} (-1)^j \big(I_\Gamma^{(p-j)}(\mathcal{K}_t) + I_\Gamma^{(p-j-1)}(\mathcal{K}_t) + H_\Gamma^{(p-j)}(\mathcal{K}_t) \\ + H_\Gamma^{(p-j-1)}(\mathcal{K}_t) - b_{p-j}\big) \end{aligned} \tag{13}$$

where b_j are the Betti numbers. For constant conformal deformations of the Riemannian metric $\mathrm{g}^{\mu\nu}$, the variation of the connected vacuum functional $\mathfrak{W}$ can be expressed in terms of the generalized zeta function $\zeta(s|\mathfrak{A})$ associated with the Laplace–Beltrami operator $\mathfrak{A}$ [36, 37]:

$$\delta\mathfrak{W} = -\zeta(0|\mathfrak{A})\log\mu^2 = (1/2)\int \mathrm{d}(\mathfrak{V}ol)\langle T_{\mu\nu}(x)\rangle \delta \mathrm{g}^{\mu\nu}(x) \tag{14}$$

where μ is a renormalization mass parameter and $\langle T_{\mu\nu}(x)\rangle$ means that all connected vacuum graphs of the stress–energy tensor $T_{\mu\nu}(x)$ are to be included. Then equation (14) leads to the result

$$\big\langle T_\mu^\mu(x)\big\rangle = \mathfrak{V}ol^{-1}\zeta(0|\mathfrak{A}) \tag{15}$$

where for $\mathbb{S}^n$: $\mathfrak{V}ol = 2\pi^{(n+1)/2}R^n/(\Gamma((n+1)/2)$, while for the compact manifold $\Gamma\backslash\mathbb{H}^n$: $\mathfrak{V}ol = \mathrm{Vol}(\Gamma\backslash G)R^n$, R being the radius corresponding to the compact space.

Our first goal is to calculate the value of generalized zeta function

$$\begin{aligned} \zeta\big(s|\mathfrak{L}_p^{(\mathrm{CE})}\big) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}\,\mathrm{Tr}\exp\big(-t\mathfrak{L}_p^{(\mathrm{CE})}\big) = \sum_{j=0}^{p} \frac{(-1)^j}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \\ \times \big(I_\Gamma^{(p-j)}(\mathcal{K}_t) + I_\Gamma^{(p-j-1)}(\mathcal{K}_t) + H_\Gamma^{(p-j)}(\mathcal{K}_t) + H_\Gamma^{(p-j-1)}(\mathcal{K}_t) - b_{p-j}\big). \end{aligned} \tag{16}$$

The integrals related to the identity contribution can be written as follows:

$$\int_0^\infty \mathrm{d}t\, t^{s-1} I_\Gamma^{(p-j)}(\mathcal{K}_t) = \frac{\chi(1)\mathrm{Vol}(\Gamma\backslash G)}{2^{2(n-1)}\Gamma(n/2)^2}\binom{n-1}{p-j}\sum_{\ell=0}^{n/2-1} a_{2\ell}^{(p-j)} \times \int_o^\infty \mathrm{d}t\, t^{s-1}\,\mathrm{e}^{-t(\alpha-j)}\int_{\mathbb{R}} \mathrm{d}r\, r^{2\ell+1}\,\mathrm{e}^{-tr^2}\tanh(\pi r) \tag{17}$$

where $\alpha \equiv p + \rho_0^2$. Using the identities

$$1 - \tanh(\pi r) = \frac{2}{1+\exp(2\pi r)} \qquad \int_0^\infty \frac{\mathrm{d}r\, r^{2\ell-1}}{1+\mathrm{e}^{2\pi r}} = \frac{(-1)^{\ell-1}}{4\ell}(1-2^{1-2\ell})B_{2\ell} \tag{18}$$

where $B_{2\ell}$ are the Bernoulli numbers, we obtain

$$\int_{\mathbb{R}} \mathrm{d}r r^{2\ell+1}\,\mathrm{e}^{-tr^2}\tanh(\pi r) = \ell! t^{-\ell-1} - \sum_{k=0}^\infty \frac{(-1)^\ell(1-2^{-2\ell-2k-1})t^k}{k!(\ell+k+1)} B_{2(\ell+k+1)} \tag{19}$$

and

$$\int_0^\infty \mathrm{d}t\, t^{s-1} I_\Gamma^{(p-j)}(\mathcal{K}_t) = \frac{\chi(1)\mathrm{Vol}(\Gamma\backslash G)}{2^{2(n-1)}\Gamma(n/2)^2}\binom{n-1}{p-j}\sum_{\ell=0}^{n/2-1} a_{2\ell}^{(p-j)} \times \left\{\frac{\ell!\Gamma(s-\ell-1)}{(\alpha-j)^{s-\ell-1}} - \sum_{k=0}^\infty \frac{(-1)^\ell(1-2^{-2\ell-2k-1}\Gamma(k+s))}{k!(\ell+k+1)}\frac{B_{2(\ell+k+1)}}{(\alpha-j)^{k+s}}\right\}. \tag{20}$$

The contribution associated with the identity integral at the point $s = 0$ becomes

$$\lim_{s\to0}\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1} I_\Gamma^{(p-j)}(\mathcal{K}_t) = \frac{\chi(1)\mathrm{Vol}(\Gamma\backslash G)}{2^{2(n-1)}\Gamma(n/2)^2}\binom{n-1}{p-j} \times \sum_{\ell=0}^{n/2-1} a_{2\ell}^{(p-j-1)}\frac{(-1)^{\ell+1}}{\ell+1}((1-2^{-2\ell-1})B_{2(\ell+1)} + [\alpha+j+1]^{\ell+1}). \tag{21}$$

The hyperbolic orbital integrals can be rewritten in terms of McDonald functions $K_\nu(z)$,

$$K_\nu(z) = 2^{-\nu-1}z^\nu\int_0^\infty \mathrm{d}t\, t^{-\nu-1}\exp(-t-z^2/(4t)) \qquad |\arg z| < \pi/2 \quad \Re z^2 > 0 \tag{22}$$

the result being

$$\int_0^\infty \mathrm{d}t\, t^{s-1} H_\Gamma^{(p-j)}(\mathcal{K}_t) = \sum_{\gamma\in C_\Gamma-\{1\}} \frac{\chi(\gamma)}{\sqrt{\pi}\, j(\gamma)} t_\gamma C(\gamma)\chi_{\sigma_{p-j}}(m_\gamma) \times \left(\frac{2\sqrt{\alpha+j}}{t_\gamma}\right)^{-s+1/2} K_{-s+1/2}(t_\gamma\sqrt{\alpha-j}). \tag{23}$$

Analysis of the integral equation (23) gives the following result (see also [14–17]): the terms associated with the hyperbolic orbital integrals vanish when $s = 0$. Finally, using equations (15), (16) and (21), we get for the trace anomaly the explicit formula

$$\langle T_\mu^\mu(x)\rangle = \frac{1}{(4\pi)^{n/2}\Gamma(n/2)R^n}\sum_{j=0}^p(-1)^j\left\{\sum_{\ell=0}^{n/2-1}\frac{(-1)^{\ell+1}}{\ell+1}\binom{n-1}{p-j} \times \left[a_{2\ell}^{(p-j)}[(1-2^{-2\ell-1})B_{2(\ell+1)} + (\alpha-j)^{\ell+1}] + a_{2\ell}^{(p-j-1)}\frac{p-j}{n-p}[(1-2^{-2\ell-1})B_{2(\ell+1)} + (\alpha-j-1)^{\ell+1}]\right]\right\} \tag{24}$$

which constitutes the main result of the present paper.

Table 1. Exact and numerical values of the anomaly for the conformally invariant scalar field, in dimensions $n = 2$ to $n = 14$ (we have set $R = 1$).

$\langle T_\mu^\mu(x)\rangle$c.i.s.	Exact	Numerical
$n = 2$	$-\frac{1}{12\pi}$	$-0.026\,5258$
$n = 4$	$-\frac{1}{240\pi^2}$	$-4.221\,72 \times 10^{-4}$
$n = 6$	$-\frac{5}{4032\pi^3}$	$-3.999\,45 \times 10^{-5}$
$n = 8$	$-\frac{23}{34\,560\pi^4}$	$-6.832\,10 \times 10^{-6}$
$n = 10$	$-\frac{263}{506\,880\pi^5}$	$-1.695\,51 \times 10^{-6}$
$n = 12$	$-\frac{133\,787}{251\,596\,800\pi^6}$	$-5.531\,07 \times 10^{-7}$
$n = 14$	$-\frac{157\,009}{232\,243\,200\pi^7}$	$-2.238\,37 \times 10^{-7}$

Table 2. Exact and numerical values of the anomaly for the family of spacetimes of dimension $n = 2$ to $n = 10$ which possess a compact spatial section, corresponding to forms of order up to $p = 4$.

$\langle T_\mu^\mu(x)\rangle$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$n = 2$	$-\frac{1}{12\pi} = -0.026\,5258$				
$n = 4$	$\frac{29}{240\pi^2} = 0.012\,243$	$-\frac{67}{160\pi^2} = -0.042\,4282$			
$n = 6$	$-\frac{1139}{4032\pi^3} = -0.009\,110\,74$	$\frac{2539}{2016\pi^3} = 0.040\,6184$	$-\frac{2005}{1792\pi^3} = -0.036\,085$		
$n = 8$	$\frac{32\,377}{345\,60\pi^4} = 0.009\,617\,53$	$-\frac{1\,368\,853}{2\,764\,80\pi^4} = -0.050\,8269$	$\frac{101\,665}{414\,72\pi^4} = 0.025\,1662$	$\frac{118\,459}{345\,60\pi^4} = 0.035\,188$	
$n = 10$	$-\frac{2\,046\,263}{5\,068\,80\pi^5} = -0.013\,1919$	$\frac{16\,454\,263}{6\,758\,40\pi^5} = 0.079\,5582$	$-\frac{2\,475\,365}{8\,110\,08\pi^5} = -0.009\,973\,89$	$-\frac{34\,196\,177}{7\,096\,320\pi^5} = -0.015\,7469$	$-\frac{14\,020\,681}{1\,351\,68\pi^5} = -0.338\,958$

The case of a conformally invariant scalar field. Restoring now the dependence on the radius R, for the specific case of a minimally coupled scalar field of mass m, we have $p = j = 0, \alpha \Rightarrow \alpha + R^2m^2$ and $\alpha = \rho_0^2$. For the case of a conformally invariant scalar field, we have $\alpha = \rho_0^2 + (n-2)R^2R(x)/[4(n-1)]$, where $R(x) = -n(n-1)R^{-2}$ is the scalar curvature. Therefore, the final result in this case becomes

$$\left\langle T_\mu^\mu(x)\right\rangle = \frac{1}{(4\pi)^{n/2}\Gamma(n/2)R^n}\sum_{\ell=0}^{n/2-1}\frac{(-1)^{\ell+1}}{\ell+1}a_{2\ell}[2^{-2\ell-2} + (1-2^{-2\ell-1})B_{2\ell+2}]. \tag{25}$$

This formula is in full agreement with a previous result obtained in [14] and constitutes a check of our main formula equation (24). In fact, we obtain from this expression table 1 (note a small misprint in the denominator of the last value given in the table in [14]).

Explicit and numerical values of the anomaly for p-forms. Using our equation (24), exact explicit values and also numerical values of the anomaly corresponding to spaces of arbitrary dimension n and forms of any order p are easily obtained, with the help of any standard program as Matlab, Maple or Mathematica. Using Mathematica 5.0 on a laptop, in a question of seconds we have obtained the following table (table 2) for the anomaly, where we have set $R = 1$ and $\alpha = p + \rho_0^2$, with $\rho_0 = (n-1)/2$.

5. Conclusions

In order to obtain information about the trace anomaly in higher dimensions, a possible and convenient way to proceed is to consider the anomaly in some specified background. Thus, the anomaly has been intensively studied for spheres. In our case, we have studied gauge theories based on Abelian p-forms on real compact hyperbolic manifolds, which are actually in the same class of constant curvature manifolds as spheres. However, this does not mean that the results are the same and, on the other hand, it is well known that the hyperbolic geometry plays a very important role in quantum theory, specifically in the theory of extended objects (string and brane theories) and cosmology. The quantum dynamic of tensor fields in hyperbolic spaces is certainly feasible and worth studying.

Generally, the trace anomaly in even dimensions contains the Euler density and a number of conformal covariant polynomials which involve the Weyl tensor and its derivatives. This general result can be integrated on conformally flat manifolds, a result which leads to the formula (15) of our paper. We have computed explicitly the trace anomaly for tensor fields on vector bundles over real compact hyperbolic spaces using the analytic continuation provided by the zeta function (or for the coefficient of the Euler density).

We have restricted ourselves to the position where the manifold is smooth and Γ is a discrete subgroup of $SO_1(n,1)$, acting freely and properly discontinuously on $\mathbb{H}^n$. The terms associated with hyperbolic orbital integrals do not contribute to the trace anomaly, as we have shown above.

Explicit exact and numerical results for the anomaly corresponding to p-forms of orders $p = 0$ to $p = 4$ in spaces of dimension $n = 2$ to $n = 10$ have been given in table 2. Both the sign and the magnitude of the anomaly seem to change in a rather non-uniform way in the cases considered, their absolute value being always less than 1 for the calculated cases (but this can be shown to be not a bound for forms of higher order). In fact, one sees clearly, that the absolute value of the conformal anomaly for p-forms definitely increases with the order of the form in hyperbolic spaces of higher dimensionality, of the class considered.

As a particular case, we recover the formula for the conformally invariant scalar field in any dimension and, in a similar way, a number of more general situations can be treated with the same techniques as those described in this paper.

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